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Material symmetry optimization by Kelvin modes

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Abstract. Pointwise optimization of the material symmetry of an anisotropic elastic material with respect to fixed and specified stress (or strain) states is accomplished. The conceptual variables in this problem are the type of material symmetry and the orientation of the canonical symmetry axis for the material at a point in the material. The actual variables are the coefficients of the elasticity (or compliance) matrix. The results are presented in the form of the elasticity (or compliance) matrices that minimize the strain energy with respect to specified, but arbitrary, stress (or strain) states.

Key words: anisotropic elasticity, Kelvin modes, optimization, strain energy

1. Introduction

In the design of plant and animal tissue it is clear that nature employs strategies that 'optimize' in some sense the microstructure of the material, and hence its material anisotropy. To see this, one has only to reflect on the grain and fibrous construction of wood and how it is loaded when in a living tree, primarily by wind. Man would like to emulate this design process for structural materials. There now exist ways to manufacture materials with specific microstructures and thus enjoy the benefits of matching the material to the details of the anticipated applied load.

We address here the question of optimizing the material symmetry of an anisotropic elastic material with respect to fixed and specified stress (or strain) states. The conceptual variables in this problem are the type of material symmetry and the orientation of the canonical symmetry axis for the material at a point in the material. The actual variables are the coefficients of the elasticity (or compliance) matrix.

When this work was done it was thought to be different from previous work in that here we tried to select both the type of material symmetry and the orientation of the canonical material symmetry axes to minimize the strain energy with respect to fixed or given design stresses (or strains). A referee brought our attention to the work of Bendsøe *et al.* [1, 2] that addresses the same problem using a different approach. The Bendsøe *et al.* work [1, 2] was done in the context of optimal structural design and has interesting points of similarity and contrast with the present work. The similarity is mainly in the statement of the problem and in some of the results. The differences are in the method of proof, global *vs*. locally posed problems, and in the imposition of cost constraints. The optimal structural design approach used in [1, 2] is a global approach that imposes constraints to account for the cost of the material employed in the design. The cost constraints, while logical, are subjective and non-unique. The present work is a local, traditional, extremum seeking, calculus approach and imposes no design constraints. The cost constraint in the global approach leads to a paradox

Table 1. The elasticity and compliance in different notations. Column 1 illustrates the Voigt notation of these quantities as fourth rank tensor components in a three-dimensional Cartesian space. Column 2 represents the Voigt matrix or double index notation. Column 3 illustrates the Kelvin-inspired notation for these quantities as second rank tensor components in a six-dimensional Cartesian space.

1	2	3	1	2	3	1	2	3	1	2	3
C_{1111}	c_{11}	$\hat{c}_{11}$	S_{1111}	s_{11}	$\hat{s}_{11}$	C_{2311}	c_{41}	$\frac{1}{\sqrt{2}}\hat{c}_{41}$	S_{2311}	$\frac{1}{2}s_{41}$	$\frac{1}{\sqrt{2}}\hat{s}_{41}$
C_{2222}	c_{22}	$\hat{c}_{22}$	S_{2222}	s_{22}	$\hat{s}_{22}$	C_{1311}	c_{51}	$\frac{1}{\sqrt{2}}\hat{c}_{51}$	S_{1311}	$\frac{1}{2}s_{51}$	$\frac{1}{\|2}\sqrt{2}\hat{s}_{51}$
C_{3333}	c_{33}	$\hat{c}_{33}$	S_{3333}	s_{33}	$\hat{s}_{33}$	C_{1211}	c_{61}	$\frac{1}{\sqrt{2}}\hat{c}_{61}$	S_{1211}	$\frac{1}{2}s_{61}$	$\frac{1}{\sqrt{2}}\hat{s}_{61}$
C_{1122}	c_{12}	$\hat{c}_{12}$	S_{1122}	s_{12}	$\hat{s}_{12}$	C_{2322}	c_{42}	$\frac{1}{\sqrt{2}}\hat{c}42$	S_{2322}	$\frac{1}{2}s_{42}$	$\frac{1}{\sqrt{2}}\hat{s}_{42}$
C_{1133}	c_{13}	$\hat{c}_{13}$	S_{1133}	s_{13}	$\hat{s}_{13}$	C_{1322}	c_{52}	$\frac{1}{\sqrt{2}}\hat{c}_{52}$	S_{1322}	$\frac{1}{2}s_{52}$	$\frac{1}{\sqrt{2}}\hat{s}_{52}$
C_{2233}	c_{23}	$\hat{c}_{23}$	S_{2233}	s_{23}	$\hat{s}_{23}$	C_{1222}	c_{62}	$\frac{1}{\sqrt{2}}\hat{c}_{62}$	S_{1222}	$\frac{1}{2}s_{62}$	$\frac{1}{\sqrt{2}}\hat{s}_{62}$
C_{2323}	c_{44}	$\frac{1}{2}\hat{c}_{44}$	S_{2323}	$\frac{1}{4}s_{44}$	$\frac{1}{2}\hat{s}_{44}$	C_{2333}	c_{43}	$\frac{1}{\sqrt{2}}\hat{c}_{43}$	S_{2333}	$\frac{1}{2}s_{43}$	$\frac{1}{\sqrt{2}}\hat{s}_{43}$
C_{1313}	c_{55}	$\frac{1}{2}\hat{c}_{55}$	S_{1313}	$\frac{1}{4}s_{55}$	$\frac{1}{2}\hat{s}_{55}$	C_{1333}	c_{53}	$\frac{1}{\sqrt{2}}\hat{c}_{53}$	S_{1333}	$\frac{1}{2}s_{53}$	$\frac{1}{\sqrt{2}}\hat{s}_{53}$
C_{1212}	c_{66}	$\frac{1}{2}\hat{c}_{66}$	S_{1212}	$\frac{1}{4}s_{66}$	$\frac{1}{2}\hat{s}_{66}$	C_{1233}	c_{63}	$\frac{1}{\sqrt{2}}\hat{c}_{63}$	S_{1233}	$\frac{1}{2}s_{63}$	$\frac{1}{\sqrt{2}}\hat{s}_{63}$
C_{1323}	c_{54}	$\frac{1}{2}\hat{c}_{54}$	S_{1323}	$\frac{1}{4}s_{54}$	$\frac{1}{2}\hat{s}_{54}$						
C_{1312}	c_{56}	$\frac{1}{2}\hat{c}_{56}$	S_{1312}	$\frac{1}{4}s_{56}$	$\frac{1}{2}\hat{s}_{56}$						
C_{1223}	c_{64}	$\frac{1}{2}\hat{c}_{64}$	S_{1223}	$\frac{1}{4}s_{64}$	$\frac{1}{2}\hat{s}_{64}$						

of [1] that requires zero shear moduli in the optimized material elasticity tensor. That difficulty is avoided in the present approach. The results and methods of [1, 2] will be compared with the present results in the text that follows and again in the discussion.

The results in this paper are obtained by a representation of the stress-strain relations due, in principle, to Kelvin [3, 4]; see [5]. In the Kelvin formulation a six-dimensional spectral representation is employed to represent the elasticity tensor or the compliance tensor. It is described in the following section. The eigenvalues and eigenvectors of the spectral representation are called the Kelvin eigenvalues or Kelvin moduli and Kelvin eigenvectors, respectively. The Kelvin eigenmodes are of two types. *Simple* Kelvin eigenmodes of a particular material symmetry contain no coefficients depending on the elastic constants, while *distributor* Kelvin eigenmodes do. The eigenmode coefficients depending on the elastic constants are called *distributors* to indicate their role in proportioning an eigenmode. It is the distributor eigenmodes that permit the proportioning of the material to the applied stress (or strain) states. The Kelvin eigenmodes and moduli for the crystalline symmetries are recorded in Appendix A.

The method employed here optimizes not only with respect to the canonical symmetry axis for the material, but also with respect to the type of material symmetry. It is the Kelvin mode matching that permits the optimization with respect to the type of material symmetry. This is described in the third section where the minimization process is accomplished by finding the elastic symmetry with the set of Kelvin modes that minimizes the energy for fixed, but arbitrary Kelvin moduli and a set of specified strain (or stress) states. In Section 4 this result is specialized to the case of finding the optimum elastic symmetry for a material that is to be subjected to one particular stress or strain state. Using these results we construct the elasticity matrix that minimizes the strain energy with respect to a single strain (stress) state in Section 5. The results are discussed in Section 6.

Table 2. Classification of the elastic coefficients for the various anisotropic symmetries.

SYMMETRY	Number of distinct constants	Number of distinct eigenvalues	Number of distributors	Number of simple eigenmodes	Number of distributor eigenmodes
Triclinic	18	6	12	0	6
Monoclinic	12	6	6	2	4
Orthotropic	9	6	3	3	3
Tetragonal	6	5	1	3	2
Trigonal	6	4	2	0	6
Hexagonal	5	4	1	2	2
Cubic	3	3	0	3	0
Isotropic	2	2	0	2	0

2. A tensorial presentation of the Kelvin formulation

The anisotropic form of Hooke's law is often written in indicial notation as $T_{ij} = C_{ijkm}E_{km}$ where the C_{ijkm} are the components of the elasticity tensor. Written as a linear transformation in six dimensions, Hooke's law has the representation $\mathbf{T} = \mathbf{cE}$ or

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix} \tag{1}$$

in the notation of Voigt [6]. The relationships of the components of C_{ijkm} to the components of the symmetric matrix $\mathbf{c}$ are given in Table 1. Introducing new notation, (1) can be rewritten in the form $\hat{\mathbf{T}} = \hat{\mathbf{c}}\hat{\mathbf{E}}$, where the shearing components of these new six-dimensional stress and strain vectors, denoted by $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$, respectively, are multiplied by $\sqrt{2}$, and $\hat{\mathbf{c}}$ is a new six-by-six matrix [7]. The matrix form of $\hat{\mathbf{T}} = \hat{\mathbf{c}}\hat{\mathbf{E}}$ is given by

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \sqrt{2}T_{23} \\ \sqrt{2}T_{13} \\ \sqrt{2}T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & c_{44} & c_{45} & c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & c_{45} & c_{55} & c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}. \tag{2}$$

The matrix $\hat{\mathbf{c}}$ is called the matrix of *elastic coefficients* and its inverse $\hat{\mathbf{s}}$, $\hat{\mathbf{E}} = \hat{\mathbf{s}}\hat{\mathbf{T}}$, $\hat{\mathbf{s}} = \hat{\mathbf{c}}^{-1}$ is called the *compliance* matrix. A chart relating these various notations for the specific elastic

coefficients is given in Table 1. The symmetric matrices $\hat{\mathbf{c}}$ and $\hat{\mathbf{s}}$ can be shown to represent the components of a second-rank tensor in a six-dimensional space, whereas the components of the matrix $\mathbf{c}$ appearing in (1) do not form a tensor [7]. The orthogonal transformation in six dimensions is represented by $\hat{\mathbf{Q}}$, which is a second-rank tensor in six dimensions that is directly related to an associated orthogonal second-rank tensor in three dimensions [7, 8]; thus, the tensor transformation law for $\hat{\mathbf{c}}$ or $\hat{\mathbf{s}}$ to a new or primed coordinate system is $\hat{\mathbf{c}}' = \hat{\mathbf{Q}}\hat{\mathbf{c}}\hat{\mathbf{Q}}^T$ or $\hat{\mathbf{s}}' = \hat{\mathbf{Q}}\hat{\mathbf{s}}\hat{\mathbf{Q}}^T$.

The eigenvalues of the matrix $\hat{\mathbf{c}}(\hat{\mathbf{s}})$ are the six numbers $\Lambda(1/\Lambda)$ satisfying the equation

$$(\hat{\mathbf{c}} - \Lambda\hat{\mathbf{1}})\hat{\mathbf{N}} = 0((\hat{\mathbf{s}} - (1/\Lambda)\hat{\mathbf{1}})\hat{\mathbf{N}} = 0), \tag{3}$$

where the vectors $\hat{\mathbf{N}}$ represent the normalized eigenvectors of $\hat{\mathbf{c}}$ (or $\hat{\mathbf{s}}$). The normalized $\hat{\mathbf{N}}$ are expressed in terms of the six-dimensional strain and stress vectors by,

$$\hat{\mathbf{E}} = \hat{\mathbf{N}}|\hat{\mathbf{E}}|, \hat{\mathbf{T}} = \hat{\mathbf{N}}|\hat{\mathbf{T}}|, |\hat{\mathbf{E}}|^2 = \hat{\mathbf{E}} \cdot \hat{\mathbf{E}}, |\hat{\mathbf{T}}|^2 = \hat{\mathbf{T}} \cdot \hat{\mathbf{T}}, \hat{\mathbf{N}} \cdot \hat{\mathbf{N}} = 1. \tag{4}$$

Since $\hat{\mathbf{c}}$ (or $\hat{\mathbf{s}}$) is positive definite, it has six positive eigenvalues. These eigenvalues are called the *Kelvin moduli* and are denoted by $\Lambda_i, i = 1, \ldots, 6$, and are ordered (if possible) by the inequalities $\Lambda_1 \geqslant \cdots \geqslant \Lambda_6 > 0$.

The eigensystems for various anisotropic elastic symmetries are described in Appendix A. Since there are, at most, six distinct eigenvalues, and since the number of distinct elastic constants exceeds six for several symmetries, the question of the role played by the other distinct elastic constants arises. These other elastic constants are called *elasticity distributors* [9, 10]. The role they play is to specify the ratios of the components for the eigenvectors $\hat{\mathbf{N}}$. Geometrically they represent the ratio of relative extensions in perpendicular directions and/or the amount of shear in an eigenmode. Since Poisson's ratios represent the ratio of relative extensions in perpendicular directions in an axial (tensile or compressive) test situation, there is some similarity between the two concepts. However, distributors are associated with eigenmodes and Poisson's ratios are associated with the axial test situation. If the eigenvector $\hat{\mathbf{N}}$ for a particular symmetry is independent of the particular value of the elastic constants for that symmetry, it is said to be a *simple* eigenmode. Thus, simple eigenmodes are independent of the distributors. If an eigenmode is not simple, it is said to be a distributor dependent eigenmode or, simply, a distributor eigenmode.

In the case of triclinic symmetry, the sum of the number of distinct eigenvalues and the number of distributors equals 18; the other three parameters to make 21 are arbitrary in the sense that they depend upon the choice of the coordinate system selected to express the elasticity tensor, see [11] or [8]. It is known that there are 18 invariants of C_{ijkm} for triclinic symmetry and Rychlewski [9] identifies one set of these invariants as the six distinct eigenvalues $\Lambda_i, i = 1, \ldots, 6$, and the twelve distributors. The twelve distributors consist of $\mathrm{tr}\mathbf{N}^{(k)}, k = 1, \ldots, 6$, and $\mathrm{tr}\mathbf{N}^{(k)}\mathbf{N}^{(k)}\mathbf{N}^{(k)}, k = 1, \ldots, 6$, where the $\mathbf{N}^{(k)}, k = 1, \ldots, 6$, are subject to the normalization conditions $\mathrm{tr}\mathbf{N}^{(k)}\mathbf{N}^{(k)} = 1, k = 1, \ldots, 6$.

The results of the preceding paragraphs above show that there exist six eigentensors of stress, denoted by $\hat{\mathbf{T}}^{(k)}, k = 1, \ldots, 6$, in the six-dimensional space, or by $\mathbf{T}^{(k)}$ in the three-dimensional space, and six eigentensors of strain, denoted by $\hat{\mathbf{E}}^{(k)}$ and $\mathbf{E}^{(k)}, k = 1, \ldots, 6$, respectively, which are related by the six equations

$$\hat{\mathbf{T}} = \Lambda_k\hat{\mathbf{E}}^{(k)}, \quad \mathbf{T} = \Lambda_k\mathbf{E}^{(k)}, \quad k = 1, \ldots, 6. \tag{5}$$

It follows that $\hat{\mathbf{c}}$ and $\hat{\mathbf{s}}$ have the representations

$$\hat{\mathbf{c}} = \sum_{k=1}^{6} \Lambda_k \hat{\mathbf{N}}^{(k)} \otimes \hat{\mathbf{N}}^{(k)}, \qquad \hat{\mathbf{s}} = \sum_{k=1}^{6} \frac{1}{\Lambda_k} \hat{\mathbf{N}}^{(k)} \otimes \hat{\mathbf{N}}^{(k)}. \tag{6}$$

The strain energy Σ,

$$\Sigma = \tfrac{1}{2}\hat{\mathbf{E}} \cdot \hat{\mathbf{c}}\hat{\mathbf{E}} = \tfrac{1}{2}\hat{\mathbf{T}} \cdot \hat{\mathbf{c}}\hat{\mathbf{T}}, \tag{7}$$

can be expressed in terms of strain, or in terms of stress, as

$$2\Sigma = \sum_{k=1}^{6} \Lambda_k |\hat{\mathbf{E}} \cdot \hat{\mathbf{N}}^{(k)}|^2 \text{ and } 2\Sigma = \sum_{k=1}^{6} \frac{1}{\Lambda_k} |\hat{\mathbf{T}} \cdot \hat{\mathbf{N}}^{(k)}|^2, \tag{8}$$

respectively; see [12]. It is important to observe the duality in $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$ on one hand, and $\hat{\mathbf{c}}$ and $\hat{\mathbf{s}}$ on the other, in this notation. A result based on the stress-strain relation, $\hat{\mathbf{T}} = \hat{\mathbf{c}}\hat{\mathbf{E}}$, is easily converted to a result based on the strain-stress relation, $\hat{\mathbf{E}} = \hat{\mathbf{s}}\hat{\mathbf{T}}$, simply by interchanging $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$ and $\hat{\mathbf{c}}$ and $\hat{\mathbf{s}}$, respectively. Thus, while the results presented in this paper optimize with respect to strain states, the results for optimization with respect to stress states are obtained simply by interchanging the following terms: $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$, $\hat{\mathbf{c}}$ and $\hat{\mathbf{s}}$, and Λ_i and $1/\Lambda_i$. Such simplicity of notation is not possible with the traditional Voigt notation.

3. The general result

The problem considered is that of finding the optimum elastic symmetry for a material that is to be subjected to certain specified stress or strain states. The optimum is determined as the symmetry that will yield the minimum strain energy. The minimization process will be accomplished by finding the elastic symmetry with the set of Kelvin modes that minimizes the energy for fixed Kelvin moduli. It is required that the strain energy be minimal for the strain states $\hat{E}_k^J$, $J = 1, 2, 3, \ldots n$. A weighing factor p_J, $p_J > 0$, is assigned to each strain state (the result for assuming stress states is exactly the same in the present notation, one only has to interchange the relevant strain and stress notations). The weighed sum of the strain energies is then given by

$$\Sigma = \frac{1}{2} \sum_{J=1}^{n} p_J c_{ij} \hat{E}_i^J \hat{E}_j^J. \tag{9}$$

Substituting (6) in (9), we have

$$\Sigma = \frac{1}{2} \sum_{k=1}^{6} \Lambda_k K_{ij} \hat{N}_i^{(k)} \hat{N}_j^{(k)}, \tag{10}$$

where the definition

$$K_{ij} = \sum_{J=1}^{n} p_J \hat{E}_i^J \hat{E}_j^J \tag{11}$$

has been introduced. The apparently innocuous transition from (9) to (10) involving the definition (11) symbolizes a different viewpoint with regard to the independent variables for the strain energy. The form (9) is conventional and suggests that the strain is the independent variable. The form (10) does not contain the strain explicitly because of the definition (11); it suggests that the Kelvin eigenmodes are the independent variables. This is appropriate because the strain (or strains) are now held fixed and the Kelvin eigenmodes are to be varied. The eigenmodes represent mode shape and they are to be 'fitted' to the fixed or 'design' strain state. An analogy might be drawn to tailoring a glove to fit a hand.

We seek to define a basis $\hat{N}_i^{(q)}$ that is optimal for the strain states $\hat{E}_k^J$, $J = 1, 2, 3, \ldots n$, weighed by the factors p_J, $p_J > 0$. In order to minimize the strain energy, an objective function ψ, constrained by the Lagrange multipliers $(1/2)\Omega^{(k)}$ contracted with the normalization constraint conditions $\hat{N}_i^{(k)}\hat{N}_i^{(k)} = 1$, is introduced:

$$\psi = \frac{1}{2}\sum_{k=1}^{6} \Lambda_k K_{ij} \hat{N}_i^{(k)} \hat{N}_j^{(k)} - \frac{1}{2}\sum_{k=1}^{6} \Omega^{(k)} (\hat{N}_q^{(k)} \hat{N}_q^{(k)} - 1). \tag{12}$$

The condition that the first derivative of (12) vanish, $\partial\psi/\partial\hat{N}_s^{(q)} = 0$, is

$$\Lambda_q K_{is} \hat{N}_i^{(q)} = \Omega^{(q)} \hat{N}_s^{(q)}, \tag{13}$$

thus

$$\Omega^{(q)} = \Lambda_q K_{im} \hat{N}_i^{(q)} \hat{N}_m^{(q)} \tag{14}$$

and

$$K_{is} \hat{N}_i^{(q)} = (K_{im} \hat{N}_i^{(q)} \hat{N}_m^{(q)}) \hat{N}_s^{(q)}. \tag{15}$$

This shows that the basis $\hat{N}_i^{(q)}$ that is optimal for the strain states $\hat{E}_k^J$, $J = 1, 2, 3, \ldots n$, weighed by the factors p_J, is given by the eigenvectors of the symmetric matrix **K** defined by (11). Inserting (14) back into the formula (10) for the strain energy, we observe that the strain energy is equal to one-half the sum of the six Lagrange multipliers $\Omega^{(k)}$,

$$\Sigma = \frac{1}{2}\sum_{k=1}^{6} \Omega^{(k)}. \tag{16}$$

Let this particular optimal basis determined from (15) be denoted by $\hat{N}_i^{(q)\#}$. To address the question of whether ψ is a maximum or a minimum at $\hat{N}_i^{(q)\#}$ we follow the guide to determining the maxima and minima of constrained functions described in [13]. First, the second derivatives of ψ are computed and evaluated at $\hat{N}_i^{(q)\#}$; thus,

$$\left(\frac{\partial^2\psi}{\partial\hat{N}_i^{(q)}\partial\hat{N}_s^{(q)}}\right)_{\hat{N}=\hat{N}^{\#}} = \Lambda_q \hat{K}_{is} - \Omega^{(q)\#}\delta_{is} \equiv \hat{G}_{is}^{(q)\#}. \tag{17}$$

The theorem of Section 89 of [13] is employed to determine if the eigenvectors of **K** produce minima or maxima of the strain energy density. The criterion for maxima or minima in the

theorem of Hancock is based on a polynomial in λ formed by setting the determinant of a 7 by 7 matrix equal to zero. The 7 by 7 matrix is constructed from $\mathbf{G}^{\#}$ given by (17) and the derivative of the normality constraint condition $(\hat{N}_i^{(k)}\hat{N}_i^{(k)} = 1)$, $\hat{N}_i^{(q)}$, evaluated at the vanishing of the first derivative, $\hat{N}_i^{(q)\#}$; thus,

$$\begin{vmatrix} G_{11}^{(q)\#}-\lambda & G_{12}^{(q)\#} & G_{13}^{(q)\#} & G_{14}^{(q)\#} & G_{15}^{(q)\#} & G_{16}^{(q)\#} & \hat{N}_1^{(q)\#} \\ G_{21}^{(q)\#} & G_{22}^{(q)\#}-\lambda & G_{23}^{(q)\#} & G_{24}^{(q)\#} & G_{25}^{(q)\#} & G_{26}^{(q)\#} & \hat{N}_2^{(q)\#} \\ G_{31}^{(q)\#} & G_{32}^{(q)\#} & G_{32}^{(q)\#}-\lambda & G_{34}^{(q)\#} & G_{35}^{(q)\#} & G_{36}^{(q)\#} & \hat{N}_3^{(q)\#} \\ G_{41}^{(q)\#} & G_{42}^{(q)\#} & G_{43}^{(q)\#} & G_{44}^{(q)\#}-\lambda & G_{45}^{(q)\#} & G_{46}^{(q)\#} & \hat{N}_4^{(q)\#} \\ G_{51}^{(q)\#} & G_{52}^{(q)\#} & G_{53}^{(q)\#} & G_{54}^{(q)\#} & G_{55}^{(q)\#}-\lambda & G_{56}^{(q)\#} & \hat{N}_5^{(q)\#} \\ G_{61}^{(q)\#} & G_{62}^{(q)\#} & G_{63}^{(q)\#} & G_{64}^{(q)\#} & G_{65}^{(q)\#} & G_{66}^{(q)\#}-\lambda & \hat{N}_6^{(q)\#} \\ \hat{N}_1^{(q)\#} & \hat{N}_2^{(q)\#} & \hat{N}_3^{(q)\#} & \hat{N}_4^{(q)\#} & \hat{N}_5^{(q)\#} & \hat{N}_6^{(q)\#} & 0 \end{vmatrix} = 0. \tag{18}$$

The theorem of Section 89 of [13] shows that ψ will have a minimum at the eigenvector $\hat{N}_i^{(q)\#}$ given by (15) if the polynomial in λ specified by the determinant (18) is invariably positive and a maximum if the polynomial is invariably negative.

From the developments presented above it is clear that the Kelvin moduli $\Lambda_i, i = 1, \ldots, 6$, are neither determined nor restricted by the analysis; only the Kelvin modes have been optimized. Thus, in the expression (6) for $\hat{\mathbf{c}}$, the six Kelvin moduli $\Lambda_i, i = 1, \ldots, 6$, are not determined. The mathematical reason for this is clear from (10) where it can be seen that the strain energy is linear in each of the eigenvalues; thus, the extrema in the strain energy due to variation in the eigenvalues are determined by the end points of their domain of definition. This domain of definition, in turn, will be determined by traditional design or manufacturing constraints. In the optimal design approach, Bendsøe *et al.* [1, 2], the cost constraints imposed are on the sum of the six Kelvin moduli or the sum of squares of the six Kelvin moduli. It is the cost constraint in terms of the six Kelvin moduli that leads to the paradox of [1] that requires zero shear moduli in the optimized material elasticity tensor.

4. Optimization for one strain (or stress) state

The problem considered is that of finding the optimum elastic symmetry for a material that is to be subjected to one particular stress or strain state. It is required that the strain energy be minimal for the strain state $\hat{E}_k^*$. The optimum basis $\hat{N}_i^{(q)}$ for this case is obtained from the expression (11) for $\mathbf{K}$ above by setting $n = 1$, $p_J = 1$, then $K_{ij} = \hat{E}_i^*\hat{E}_j^*$ and placing this representation in (15) we obtain

$$\hat{E}_s^*(\hat{\mathbf{E}}^* \cdot \hat{\mathbf{N}}^{(q)}) = (\hat{\mathbf{E}}^* \cdot \hat{\mathbf{N}}^{(q)})^2 \hat{N}_s^{(q)}. \tag{19}$$

This result shows that, if $\hat{\mathbf{E}}^* \cdot \hat{\mathbf{N}}^{(q)} \neq 0$, then

$$\hat{E}_s^* = (\hat{\mathbf{E}}^* \cdot \hat{\mathbf{N}}^{(q)})\hat{N}_s^{(q)}, \tag{20}$$

and it follows that a basis solution $\hat{N}_i^{(q)}$ to (19) is that one mode, say $\hat{N}_i^{(1)}$ coincide with the normalized form of the strain state $\hat{E}_k^*$,

$$\hat{N}_i^{(1)} = \frac{\hat{E}_i^*}{\|\hat{E}^*\|}, \tag{21}$$

and the other five $\hat{N}_i^{(q)}$, $q = 2, 3, 4, 5, 6$ be perpendicular to it,

$$\hat{\mathbf{E}}^* \cdot \hat{\mathbf{N}}^{(q)} = 0, q = 2, 3, 4, 5, 6. \tag{22}$$

It follows then from (14) that the Lagrange multipliers are given by

$$\Omega^{(1)} = \Lambda_1 \|\hat{\mathbf{E}}^*\|^2, \Omega^{(q)} = 0, q = 2, 3, 4, 5, 6, \tag{23}$$

and, thus, from (16), the strain energy associated with the strain state $\hat{\mathbf{E}}^*$ is given by the simple formula

$$\Sigma = \frac{\Lambda_1}{2} \|\hat{\mathbf{E}}^*\|^2. \tag{24}$$

This formula is the basis of our use of the descriptive phrase 'fitting like a glove' to characterize the relationship between the specified strain state $\hat{\mathbf{E}}^*$ and the elasticity matrix $\hat{\mathbf{c}}$. The formula shows that the strain energy depends upon the strain through only one unspecified eigenvalue, Λ_1. The result shows clearly that only one Kelvin mode is involved. This Kelvin mode has been tailored like a glove to fit the strain state $\hat{\mathbf{E}}^*$ that was optimized for, and it does.

The second derivatives of the objective function ψ evaluated at the value of $\hat{N}_i^{(q)}$ for which the first derivative vanishes are

$$\begin{aligned} \hat{G}_{is}^{(1)\#} &= \Lambda_1(\hat{E}_i^* \hat{E}_s^* - \|\hat{\mathbf{E}}^*\|\delta_{is}), \\ \hat{G}_{is}^{(q)\#} &= \Lambda_q \hat{E}_i^* \hat{E}_i^*, \quad q = 2, 3, 4, 5, 6. \end{aligned} \tag{25}$$

When we substituted $\hat{G}_{is}^{(1)\#}$ in (18) and determined the polynomial in λ by subsequently taking the determinant of the result, we find that the solution $\hat{N}_i^{(1)\#}$ corresponds to a minimum in the objective function ψ, since the resulting polynomial in λ,

$$(\|\hat{\mathbf{E}}^*\|^2 + \lambda)^5 = 0 \tag{26}$$

is invariably positive. The solutions $\hat{N}_i^{(q)\#}$, $q = 2, 3, 4, 5, 6$, all correspond to zero strain energy modes.

These results are employed in the following section to obtain the explicit form of $\hat{\mathbf{c}}$ that is fitted like a glove to the single specified strain state $\hat{\mathbf{E}}^*$.

5. The elasticity matrix fitted to strain (stress) state

We will construct the elasticity matrix corresponding to the basis $\hat{N}_i^{(q)}$ that minimizes the strain energy with respect to a single strain (stress) state using the results of the previous

section. We consider first the case in which all three principal strains of the strain state $\hat{\mathbf{E}}^*$ are distinct.

(a) All three principal strains of the strain state $\hat{\mathbf{E}}^*$ are distinct.

The fact that all three principal strains of the strain state $\hat{\mathbf{E}}^*$ are distinct limits the possible elastic symmetries that are compatible. The only symmetries that admit a basis $\hat{N}_i^{(q)}$ containing distributor eigenmodes with three components whose values are unrelated are orthotropic, monoclinic and triclinic symmetry, see [7]. We select the greatest of these symmetries, orthotropy, for this example. There would be more degrees of freedom available if monoclinic or triclinic symmetry were selected; the advantage of orthotropic symmetric is the lesser number of degrees of freedom. The three-dimensional coordinate system of the orthotropic symmetry is taken as the three-dimensional principal coordinate system of the strain tensor $\mathbf{E}^*$. The six-dimension vector $\hat{\mathbf{E}}^*$ is represented in the six-dimensional coordinate system corresponding to its three-dimensional principal coordinate system; in this six-dimensional coordinate system its components are $(\hat{E}_1^*, \hat{E}_2^*, \hat{E}_3^*, 0, 0, 0)$. Normalizing this form of $\hat{\mathbf{E}}^*$ and using (21) we obtain the first element of the basis,

$$\hat{N}_i^{(1)} = \frac{\hat{E}_i^*}{\|\hat{\mathbf{E}}^*\|}, \qquad \hat{\mathbf{N}}^{(1)} = \frac{(\hat{E}_1^*, \hat{E}_2^*, \hat{E}_3^*, 0, 0, 0)}{\sqrt{(\hat{E}_1^*)^2 + (\hat{E}_2^*)^2 + (\hat{E}_3^*)^2}}, \tag{27}$$

and the other five elements of the basis must satisfy the orthogonality condition (22). To simplify notation, the components of $\hat{N}_i^{(1)}$ are expressed in terms of two angles, α and β, thus

$$\begin{aligned} \hat{N}_1^{(1)} &= \frac{\hat{E}_1^*}{\|\hat{\mathbf{E}}^*\|} \equiv \sin\beta\cos\alpha, \qquad \hat{N}_2^{(1)} = \frac{\hat{E}_2^*}{\|\hat{\mathbf{E}}^*\|} \equiv -\sin\beta\sin\alpha, \\ \hat{N}_3^{(1)} &= \frac{\hat{E}_3^*}{\|\hat{\mathbf{E}}^*\|} \equiv \cos\beta, \qquad \hat{N}_4^{(1)} = \hat{N}_5^{(1)} = \hat{N}_6^{(1)} = 0. \end{aligned} \tag{28}$$

Consistent with the selected orthotropic symmetry the other base vectors, orthogonal to $\hat{N}_i^{(1)}$, are given by

$$\begin{aligned} &\hat{N}_1^{(2)} = \cos\alpha\cos\beta\cos\theta + \sin\alpha\sin\theta, \\ &\hat{N}_2^{(2)} = -\sin\alpha\cos\beta\cos\theta + \cos\alpha\sin\theta, \quad \hat{N}_3^{(2)} = -\sin\beta\cos\theta, \\ &\hat{N}_4^{(2)} = \hat{N}_5^{(2)} = \hat{N}_6^{(2)} = 0, \\ &\hat{N}_1^{(3)} = -\cos\alpha\cos\beta\sin\theta + \sin\alpha\cos\theta, \\ &\hat{N}_2^{(3)} = \sin\alpha\cos\beta\sin\theta + \cos\alpha\sin\theta, \\ &\hat{N}_3^{(3)} = \sin\beta\sin\theta, \quad \hat{N}_4^{(3)} = \hat{N}_5^{(3)} = \hat{N}_6^{(3)} = 0, \\ &\hat{N}_1^{(4)} = 0, \hat{N}_2^{(4)} = 0, \hat{N}_3^{(4)} = 0, \hat{N}_4^{(4)} = 1, \hat{N}_5^{(4)} = 0, \hat{N}_6^{(4)} = 0, \\ &\hat{N}_1^{(5)} = 0, \hat{N}_2^{(5)} = 0, \hat{N}_3^{(5)} = 0, \hat{N}_4^{(5)} = 0, \hat{N}_5^{(5)} = 1, \hat{N}_6^{(5)} = 0, \\ &\hat{N}_1^{(6)} = 0, \hat{N}_2^{(6)} = 0, \hat{N}_3^{(6)} = 0, \hat{N}_4^{(6)} = 0, \hat{N}_5^{(6)} = 0, \hat{N}_6^{(6)} = 1, \end{aligned} \tag{29}$$

where the parameter θ represents an arbitrary rotation about the axis $\hat{N}_i^{(1)}$, arbitrary in the sense that it is not specified by the strain state $\hat{\mathbf{E}}^*$. The parameter θ represents an unspecified

distributor; it and the six Kelvin moduli Λ_i, $i = 1, \ldots, 6$, are the seven parameters appearing in the expression for $\hat{\mathbf{c}}$ that are not determined by the strain state $\hat{\mathbf{E}}^*$. Since orthotropic symmetry has nine elastic coefficients, six Kelvin moduli and three distributors, only two of these parameters are specified by the optimization with respect to the strain state $\hat{\mathbf{E}}^*$. The seven parameters, θ and the six Kelvin moduli Λ_i, $i = 1, \ldots, 6$, are still arbitrary.

Substituting the basis (29) in (6) we obtain the following expression for $\hat{\mathbf{c}}$, in terms of the strain state parameters α and β and the unspecified arbitrary parameters, θ and the six Kelvin moduli Λ_i, $i = 1, \ldots, 6$:

$$\hat{\mathbf{c}} = \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} & 0 & 0 & 0 \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}_{66} \end{bmatrix}, \tag{30}$$

where

$$\begin{aligned}
\hat{c}_{11} &= \Lambda_1(\cos\alpha \sin\beta)^2 + \Lambda_2(\cos\alpha \cos\beta \cos\theta + \sin\alpha \sin\theta)^2 \\
&\quad +\Lambda_3(\sin\alpha \cos\theta - \cos\alpha \cos\beta \sin\theta)^2, \\
\hat{c}_{22} &= \Lambda_1(\sin\alpha \sin\beta)^2 + \Lambda_2(\cos\alpha \sin\theta - \sin\alpha \cos\beta \cos\theta)^2 \\
&\quad +\Lambda_3(\cos\alpha \cos\theta + \sin\alpha \cos\beta \sin\theta)^2, \\
\hat{c}_{33} &= \Lambda_1(\cos\beta)^2 + \Lambda_2(\sin\beta \cos\theta)^2 + \Lambda_3(\sin\beta \sin\theta)^2, \\
\hat{c}_{23} &= -\Lambda_1(\cos\beta \sin\beta \sin\alpha) - \Lambda_2 \cos\theta \sin\beta(\cos\alpha \sin\theta - \sin\alpha \cos\beta \cos\theta) \\
&\quad +\Lambda_3 \sin\beta \cos\theta(\cos\alpha \cos\theta + \sin\alpha \cos\beta \sin\theta), \\
\hat{c}_{13} &= \Lambda_1(\cos\beta \sin\beta \cos\alpha) - \Lambda_2 \cos\theta \sin\beta(\sin\alpha \sin\theta + \cos\alpha \cos\beta \cos\theta) \\
&\quad +\Lambda_3 \sin\beta \sin\theta(\sin\alpha \cos\theta - \cos\alpha \cos\beta \sin\theta), \\
\hat{c}_{12} &= -\Lambda_1 \sin\alpha \cos\alpha(\sin\beta)^2 \\
&\quad +\Lambda_2(\sin\alpha \sin\theta + \cos\alpha \cos\beta \cos\theta)(\cos\alpha \sin\theta - \sin\alpha \cos\beta \cos\theta) \\
&\quad +\Lambda_3(\sin\alpha \cos\theta - \cos\alpha \cos\beta \cos\theta)(\cos\alpha \cos\theta + \sin\alpha \cos\beta \sin\theta), \\
\hat{c}_{44} &= \Lambda_4, \hat{c}_{55} = \Lambda_5, \hat{c}_{66} = \Lambda_6.
\end{aligned} \tag{31}$$

(b) Two of the three principal strains of the strain state $\hat{\mathbf{E}}^*$ are equal.

The symmetries that admit a basis $\hat{N}_i^{(q)}$ containing eigenvectors with two equal, and one distinct, dilatational eigenmodes are tetragonal, trigonal, hexagonal (transverse isotropy), orthotropic, monoclinic and triclinic symmetry. We select the three greatest of these symmetries, tetragonal, trigonal and hexagonal (transverse isotropy) for illustration. The eigenbases for tetragonal and hexagonal (transverse isotropy) symmetries are coincident [7] and will be considered first. These symmetries have the same eigenbases and are only distinguished by different Kelvin moduli. The three-dimensional canonical material symmetry coordinate system is taken as the three-dimensional principal coordinate system of the strain tensor $\mathbf{E}^*$.

The results described above for the case of three distinct principal strains is specialized to the present situation by taking $\alpha = -\pi/4$ and $\theta = 0$. Thus, the basis for tetragonal and/or hexagonal (transverse isotropy) is given by

$$\begin{aligned}
\hat{N}_1^{(1)} &= \hat{N}_2^{(1)} = \frac{\sin\beta}{\sqrt{2}}, \quad \hat{N}_3^{(1)} = \cos\beta, \quad \hat{N}_4^{(1)} = 0, \quad \hat{N}_5^{(1)} = 0, \quad \hat{N}_6^{(1)} = 0,\\
\hat{N}_1^{(2)} &= \hat{N}_2^{(2)} = \frac{\cos\beta}{\sqrt{2}}, \quad \hat{N}_3^{(2)} = -\sin\beta, \quad \hat{N}_4^{(2)} = 0, \quad \hat{N}_5^{(2)} = 0, \quad \hat{N}_6^{(2)} = 0,\\
\hat{N}_1^{(3)} &= -\hat{N}_2^{(3)} = \frac{1}{\sqrt{2}}, \quad \hat{N}_3^{(3)} = 0, \quad \hat{N}_4^{(3)} = 0, \quad \hat{N}_5^{(3)} = 0, \quad \hat{N}_6^{(3)} = 0,\\
\hat{N}_1^{(4)} &= 0, \quad \hat{N}_2^{(4)} = 0, \quad \hat{N}_3^{(4)} = 0, \quad \hat{N}_4^{(4)} = 1, \quad \hat{N}_5^{(4)} = 0, \quad \hat{N}_6^{(4)} = 0,\\
\hat{N}_1^{(5)} &= 0, \quad \hat{N}_2^{(5)} = 0, \quad \hat{N}_3^{(5)} = 0, \quad \hat{N}_4^{(5)} = 0, \quad \hat{N}_5^{(5)} = 0, \quad \hat{N}_6^{(5)} = 0,\\
\hat{N}_1^{(6)} &= 0, \quad \hat{N}_2^{(6)} = 0, \quad \hat{N}_3^{(6)} = 0, \quad \hat{N}_4^{(6)} = 0, \quad \hat{N}_5^{(6)} = 0, \quad \hat{N}_6^{(6)} = 1.
\end{aligned} \tag{32}$$

Substituting the basis (32) in (8) we have the following expression for the tetragonal symmetry $\hat{\mathbf{c}}$, in terms of the strain state parameter β and the unspecified Kelvin moduli Λ_i, $i = 1, \ldots, 6$,

$$\hat{\mathbf{c}} = \begin{bmatrix}
\hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & 0\\
\hat{c}_{12} & \hat{c}_{11} & \hat{c}_{13} & 0 & 0 & 0\\
\hat{c}_{13} & \hat{c}_{13} & \hat{c}_{33} & 0 & 0 & 0\\
0 & 0 & 0 & \hat{c}_{44} & 0 & 0\\
0 & 0 & 0 & 0 & \hat{c}_{44} & 0\\
0 & 0 & 0 & 0 & 0 & \hat{c}_{66}
\end{bmatrix}, \tag{33}$$

where

$$\begin{aligned}
\hat{c}_{11} &= \hat{c}_{22} = \tfrac{1}{2}(\Lambda_1(\sin\beta)^2 + \Lambda_2(\cos\beta)^2 + \Lambda_3),\\
\hat{c}_{33} &= \Lambda_1(\cos\beta)^2 + \Lambda_2(\sin\beta)^2,\\
\hat{c}_{12} &= \tfrac{1}{2}(\Lambda_1(\sin\beta)^2 + \Lambda_2(\cos\beta)^2 - \Lambda_3),\\
\hat{c}_{13} &= \frac{\sin 2\beta(\Lambda_1 - \Lambda_2)}{2\sqrt{2}}, \quad \hat{c}_{44} = \hat{c}_{55} = \Lambda_4, \quad \hat{c}_{66} = \Lambda_5.
\end{aligned} \tag{34}$$

The expression (33) for the tetragonal symmetry $\hat{\mathbf{c}}$ reduces to that for hexagonal (transverse isotropy) symmetry when $\hat{c}_{66} = \hat{c}_{11} - \hat{c}_{12}$, or equivalently, $\Lambda_5, = \Lambda_3$.

We turn now to the case of trigonal symmetry, which also admits a basis $\hat{N}_i^{(q)}$ containing eigenvectors with two equal, and one distinct, dilatational eigenmodes. The first two eigenvectors in the basis for trigonal symmetry coincide with those for hexagonal symmetry; that is the first two lines of (32). The other four depend upon an unspecified distributor here denoted by γ,

$$\begin{aligned}
\hat{N}_1^{(3)} &= -\hat{N}_2^{(3)} = \frac{\cos\gamma}{2}, \quad \hat{N}_3^{(3)} = 0,\\
\hat{N}_4^{(3)} &= \sin\gamma, \quad \hat{N}_5^{(3)} = 0, \quad \hat{N}_6^{(3)} = 0,
\end{aligned}$$

$$
\begin{aligned}
\hat{N}_1^{(4)} &= -\hat{N}_2^{(4)} = -\frac{\sin\gamma}{2}, \quad \hat{N}_3^{(4)} = 0, \quad \hat{N}_4^{(4)} = \sin\gamma, \\
\hat{N}_5^{(4)} &= 0, \quad \hat{N}_6^{(4)} = 0, \\
\hat{N}_1^{(5)} &= 0, \quad \hat{N}_2^{(5)} = 0, \quad \hat{N}_3^{(5)} = 0, \quad \hat{N}_4^{(5)} = 0, \\
\hat{N}_5^{(5)} &= \cos\gamma, \quad \hat{N}_6^{(5)} = -\sin\gamma, \\
\hat{N}_1^{(6)} &= 0, \quad \hat{N}_2^{(6)} = 0, \quad \hat{N}_3^{(6)} = 0, \quad \hat{N}_4^{(6)} = 0, \\
\hat{N}_5^{(6)} &= \sin\gamma, \quad \hat{N}_6^{(6)} = \cos\gamma.
\end{aligned}
\tag{35}
$$

Substituting the basis (35) in (8) we arrive at the following expression for the trigonal symmetry $\hat{c}$, in terms of the strain state parameter β, the unspecified distributor parameter γ and the unspecified Kelvin moduli $\Lambda_i, i = 1, \ldots, 6$,

$$
\hat{\mathbf{c}} = \begin{bmatrix}
\hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & \hat{c}_{14} & 0 & 0 \\
\hat{c}_{12} & \hat{c}_{11} & \hat{c}_{13} & -\hat{c}_{14} & 0 & 0 \\
\hat{c}_{13} & \hat{c}_{13} & \hat{c}_{33} & 0 & 0 & 0 \\
\hat{c}_{14} & -\hat{c}_{14} & 0 & \hat{c}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \hat{c}_{44} & \sqrt{2}\hat{c}_{14} \\
0 & 0 & 0 & 0 & \sqrt{2}\hat{c}_{14} & \hat{c}_{11} - \hat{c}_{12}
\end{bmatrix},
\tag{36}
$$

where

$$
\begin{aligned}
\hat{c}_{11} &= \tfrac{1}{2}(\Lambda_1(\sin\beta)^2 + \Lambda_2(\cos\beta)^2 + \tfrac{1}{2}(\Lambda_3(\cos\gamma)^2 + \Lambda_4(\sin\gamma)^2)), \\
\hat{c}_{12} &= \tfrac{1}{2}(\Lambda_1(\sin\beta)^2 + \Lambda_2(\cos\beta)^2 - \tfrac{1}{2}(\Lambda_3(\cos\gamma)^2 + \Lambda_4(\sin\gamma)^2)), \\
\hat{c}_{33} &= \Lambda_1(\sin\beta)^2 + \Lambda_2(\cos\beta)^2, \quad \hat{c}_{13} = \frac{\sin 2\beta(\Lambda_1 - \Lambda_2)}{2\sqrt{2}}, \\
\hat{c}_{44} &= \Lambda_3(\cos\gamma)^2 + \Lambda_4(\sin\gamma)^2, \quad \hat{c}_{14} = \frac{\sin 2\beta(\Lambda_3 - \Lambda_4)}{4}.
\end{aligned}
\tag{37}
$$

(c) The three principal strains of the strain state $\hat{\mathbf{E}}^*$ are equal.

A strain state with three equal principal strains requires eigenmodes with three equal components under the optimization process above. All symmetries admit a basis $\hat{N}_i^{(q)}$ containing eigenvectors with three equal components. Isotropic and cubic symmetry contain simple eigenmodes with three equal components. The eigenvectors for isotropic and cubic symmetry are given in [7]. For lesser symmetries we may adjust the distributor eigenmodes to accommodate a strain state with three equal principal strains using the methods described above for the other cases.

6. Discussion

The results outlined above represent an attempt to understand the effect of adaptive anisotropy mechanisms that function in many materials. In natural materials such as plant and animal

tissues and in man-made composites, the adaptive anisotropy mechanisms stem from active sources. In geological materials and in ductile structural materials the adaptive anisotropy mechanisms are due to passive, or reactive mechanisms. These ideas are considered in greater detail in the discussion section of [14]. In the present paper we reported the pointwise optimization of the material symmetry of an anisotropic elastic material with respect to fixed and specified stress (or strain) states. The results were presented in the form of the elasticity (or compliance) matrices that minimize the strain energy with respect to specified, but arbitrary, stress (or strain) states.

These results may be extended in several directions not explicitly developed in the paper. For example, in the case of designing a material symmetry for a strain state with three distinct principal strains it was shown that a material with orthotropic symmetry satisfied the minimization criterion by specifying only two parameters influencing the nine distinct elastic constants. This result may be extended to the case of designing a material symmetry for two strain states with three distinct principal strains, but coincident principal directions, by specifying three parameters influencing the nine distinct elastic constants. In the case when only two of the three principal strains of the design strain state are distinct, the trigonal symmetry solution had an additional degree of distributor freedom that may be used in a similar way to accommodate an additional strain state.

However, a typical solution to the general case outlined in Section 3 must be carried out numerically. The matrix **K** defined by (11) is calculated from the specified strain states $\hat{E}_k^J$ and the weighing factor is p_J, $p_J > 0$, $J = 1, 2, 3, \ldots n$, assigned to each strain state. The eigenvectors of this matrix are then calculated and compared with the form of the eigenvectors for distinct anisotropic linear elastic symmetries (see [7]). The type of elastic symmetry and the orientation of the canonical symmetry axis are determined by this calculation. The Kelvin moduli, Λ_i, $i = 1, \ldots, n$, $n \leqslant 6$, and the unspecified distributors (if any) must be determined by other design and/or manufacturing constraints.

In this regard, a comparison with the optimal structural design approach used in Bendsøe *et al.* [1, 2] is informative. While our result is a straightforward calculus problem, the global optimal structural design approach attempts to introduce real world constraints into the formulated problem. These cost constraints, while logical, are subjective and non-unique. Other cost constraints will lead to other features in the form of the elasticity tensor of the optimized material. It is possible that the paradox of [1] that required zero shear moduli in the optimized material elasticity tensor could be avoided with the selection of a different constraint.

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Appendix A. Eigenmodes for the various linear elastic symmetries

In this section the eigenmodes of the various linear elastic symmetries will be summarized. The modes were determined by Kelvin [3], Rychlewski [9], Mehrabadi and Cowin [5, 7] and Cowin and Mehrabadi [10].

We begin with the eigenmodes for cubic symmetry, and note that those for isotropic symmetry are a special case. The eigenproblem (4) is expressed for cubic symmetry by using the representation for $\hat{\mathbf{c}}$ in a cubic symmetry coordinate system; thus,

$$(\hat{\mathbf{c}}^{\mathrm{Cub}} - \Lambda\mathbf{1})\hat{\mathbf{N}} = \begin{bmatrix} \hat{c}^{\mathrm{Cub}}_{11} - \Lambda & \hat{c}^{\mathrm{Cub}}_{12} & \hat{c}^{\mathrm{Cub}}_{12} & 0 & 0 & 0 \\ \hat{c}^{\mathrm{Cub}}_{12} & \hat{c}^{\mathrm{Cub}}_{11} - \Lambda & \hat{c}^{\mathrm{Cub}}_{12} & 0 & 0 & 0 \\ \hat{c}^{\mathrm{Cub}}_{12} & \hat{c}^{\mathrm{Cub}}_{12} & \hat{c}^{\mathrm{Cub}}_{11} - \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}^{\mathrm{Cub}}_{44} - \Lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}^{\mathrm{Cub}}_{44} - \Lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}^{\mathrm{Cub}}_{44} - \Lambda \end{bmatrix} \times \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \\ \hat{N}_3 \\ \hat{N}_4 \\ \hat{N}_5 \\ \hat{N}_6 \end{bmatrix} = \mathbf{0}. \tag{A1}$$

The eigenvalues of (A1) are of multiplicity one, two and three and are given by

$$\Lambda^{\mathrm{Cub}}_{(1)} = \hat{c}^{\mathrm{Cub}}_{11} + \hat{c}^{\mathrm{Cub}}_{12}, \quad \Lambda^{\mathrm{Cub}}_{(2,3)} = \hat{c}^{\mathrm{Cub}}_{11} - \hat{c}^{\mathrm{Cub}}_{12}, \quad \Lambda^{\mathrm{Cub}}_{(4,5,6)} = \hat{c}^{\mathrm{Cub}}_{44}$$

or (A2)

$$\Lambda^{\mathrm{Cub}}_{(1)} = c^{\mathrm{Cub}}_{11} + c^{\mathrm{Cub}}_{12}, \quad \Lambda^{\mathrm{Cub}}_{(2,3)} = c^{\mathrm{Cub}}_{11} - c^{\mathrm{Cub}}_{12}, \quad \Lambda^{\mathrm{Cub}}_{(4,5,6)} = 2c^{\mathrm{Cub}}_{44},$$

respectively, and a set of eigentensors of (A1) corresponding to these eigenvalues is

$$\begin{aligned} \mathbf{N}^{(1)\mathrm{Cub}} &= \frac{1}{\sqrt{3}}\mathbf{1}, \quad \mathbf{N}^{(2)\mathrm{Cub}} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \\ \mathbf{N}^{(3)\mathrm{Cub}} &= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}^{(4)\mathrm{Cub}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\ \mathbf{N}^{(5)\mathrm{Cub}} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}^{(6)\mathrm{Cub}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{A3}$$

respectively. Note that these eigentensors are independent of the elastic constants and therefore reflect simple eigenmodes. These results for cubic symmetry readily reduce to those for isotropic symmetry when $\hat{c}^{\mathrm{Cub}}_{44} = \hat{c}^{\mathrm{Cub}}_{11} - \hat{c}^{\mathrm{Cub}}_{12}$ and Cub may be replaced by Iso in (A1) through (A3). The eigenvalues for isotropic symmetry are of multiplicity one and five and are given by

$$\Lambda^{\mathrm{Iso}}_{(1)} = \hat{c}^{\mathrm{Iso}}_{11} + 2\hat{c}^{\mathrm{Iso}}_{12}, \quad \Lambda^{\mathrm{Iso}}_{(2,3,4,5,6)} = \hat{c}^{\mathrm{Iso}}_{11} - \hat{c}^{\mathrm{Iso}}_{12} = \hat{c}^{\mathrm{Iso}}_{44}. \tag{A4}$$

The eigenproblem (3) is expressed for tetragonal symmetry by using the representation for $\hat{\mathbf{c}}$ in a tetragonal symmetry coordinate system; thus,

$$(\hat{\mathbf{c}}^{\mathrm{Tet}} - \Lambda\mathbf{1})\hat{\mathbf{N}} = \begin{bmatrix} \hat{c}^{\mathrm{Tet}}_{11} - \Lambda & \hat{c}^{\mathrm{Tet}}_{12} & \hat{c}^{\mathrm{Tet}}_{13} & 0 & 0 & 0 \\ \hat{c}^{\mathrm{Tet}}_{12} & \hat{c}^{\mathrm{Tet}}_{11} - \Lambda & \hat{c}^{\mathrm{Tet}}_{13} & 0 & 0 & 0 \\ \hat{c}^{\mathrm{Tet}}_{13} & \hat{c}^{\mathrm{Tet}}_{13} & \hat{c}^{\mathrm{Tet}}_{11} - \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}^{\mathrm{Tet}}_{44} - \Lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}^{\mathrm{Tet}}_{44} - \Lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}^{\mathrm{Tet}}_{44} - \Lambda \end{bmatrix} \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \\ \hat{N}_3 \\ \hat{N}_4 \\ \hat{N}_5 \\ \hat{N}_6 \end{bmatrix} = \mathbf{0}. \tag{A5}$$

The eigenvalues of (A5) are of multiplicity one, one, one, two and one and are given by

$$\Lambda^{\mathrm{Tet}}_{(1)}, \quad \Lambda^{\mathrm{Tet}}_{(2)} = \tfrac{1}{2}\left[(\hat{c}^{\mathrm{Tet}}_{11} + \hat{c}^{\mathrm{Tet}}_{12} + \hat{c}^{\mathrm{TeT}}_{33}) \pm \sqrt{8(\hat{c}^{\mathrm{Tet}}_{13})^2 + (\hat{c}^{\mathrm{Tet}}_{11} + \hat{c}^{\mathrm{Tet}}_{12} - \hat{c}^{\mathrm{Tet}}_{33})^2}\right],$$
$$\Lambda^{\mathrm{Tet}}_{(3)} = \hat{c}^{\mathrm{Tet}}_{11} - \hat{c}^{\mathrm{Tet}}_{12}, \quad \Lambda^{\mathrm{Tet}}_{(4,5)} = \hat{c}^{\mathrm{Tet}}_{44}, \quad \Lambda^{\mathrm{Tet}}_{(6)} = \hat{c}^{\mathrm{Tet}}_{66}, \tag{A6}$$

respectively, and the first three eigentensors corresponding to the first three eigenvalues are

$$\mathbf{N}^{(1)\mathrm{Tet}} = \begin{bmatrix} \frac{1}{2}(\cos\alpha + \sin\alpha) & 0 & 0 \\ 0 & \frac{1}{2}(\cos\alpha + \sin\alpha) & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha) \end{bmatrix},$$
$$\mathbf{N}^{(2)\mathrm{Tet}} = \begin{bmatrix} \frac{1}{2}(\sin\alpha - \cos\alpha) & 0 & 0 \\ 0 & \frac{1}{2}(\sin\alpha - \cos\alpha) & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha) \end{bmatrix}, \tag{A7}$$
$$\mathbf{N}^{(3)\mathrm{Tet}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively, where

$$\tan 2\alpha = \frac{(\hat{c}^{\mathrm{Tet}}_{11} + \hat{c}^{\mathrm{Tet}}_{12} - \hat{c}^{\mathrm{Tet}}_{33})}{2\sqrt{2}\hat{c}^{\mathrm{Tet}}_{13}}. \tag{A8}$$

The first two eigentensors are associated with distributor eigenmodes because of the dependence upon the elastic constants through the angle α. This definition of the angle α is twice the angle α defined by Mehrabadi and Cowin [7] in their Equation (5.11). The eigentensors $\mathbf{N}^{(4)\mathrm{Tet}}$, $\mathbf{N}^{(5)\mathrm{Tet}}$ and $\mathbf{N}^{(6)\mathrm{Tet}}$ coincide with $\mathbf{N}^{(4)\mathrm{Cub}}$, $\mathbf{N}^{(5)\mathrm{Cub}}$ and $\mathbf{N}^{(6)\mathrm{Cub}}$ given by (A3).

In the case of trigonal symmetry the eigenproblem (3) is expressed by using the representation for $\hat{\mathbf{c}}$ in a trigonal symmetry coordinate system, thus

$$(\hat{\mathbf{c}}^{\mathrm{Tri}} - \Lambda\mathbf{1})\hat{\mathbf{N}} = \begin{bmatrix} \hat{c}^{\mathrm{Tri}}_{11} - \Lambda & \hat{c}^{\mathrm{Tri}}_{12} & \hat{c}^{\mathrm{Tri}}_{13} & \hat{c}^{\mathrm{Tri}}_{14} & 0 & 0 \\ \hat{c}^{\mathrm{Tri}}_{12} & \hat{c}^{\mathrm{Tri}}_{11} - \Lambda & \hat{c}^{\mathrm{Tri}}_{13} & -\hat{c}^{\mathrm{Tri}}_{14} & 0 & 0 \\ \hat{c}^{\mathrm{Tri}}_{13} & \hat{c}^{\mathrm{Tri}}_{13} & \hat{c}^{\mathrm{Tri}}_{33} - \Lambda & 0 & 0 & 0 \\ \hat{c}^{\mathrm{Tri}}_{14} & -\hat{c}^{\mathrm{Tri}}_{14} & 0 & \hat{c}^{\mathrm{Tri}}_{13} - \Lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}^{\mathrm{Tri}}_{44} - \Lambda & \sqrt{2}\hat{c}^{\mathrm{Tri}}_{14} \\ 0 & 0 & 0 & 0 & \sqrt{2}\hat{c}^{\mathrm{Tri}}_{14} & \hat{c}^{\mathrm{Tri}}_{11} - \hat{c}^{\mathrm{Tri}}_{12} - \Lambda \end{bmatrix} \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \\ \hat{N}_3 \\ \hat{N}_4 \\ \hat{N}_5 \\ \hat{N}_6 \end{bmatrix} = \mathbf{0}. \tag{A9}$$

The eigenvalues of (A9) are of multiplicity one, one, two and two and are given by

$$\begin{aligned}
\Lambda^{\mathrm{Tri}}_{(1)} &= \tfrac{1}{2}\left[(\hat{c}^{\mathrm{Tri}}_{11}+\hat{c}^{\mathrm{Tri}}_{12}+\hat{c}^{\mathrm{Tri}}_{33}) + \sqrt{8(\hat{c}^{\mathrm{Tri}}_{13})^2+(\hat{c}^{\mathrm{Tri}}_{11}+\hat{c}^{\mathrm{Tri}}_{12}-\hat{c}^{\mathrm{Tri}}_{33})^2}\right],\\
\Lambda^{\mathrm{Tri}}_{(2)} &= \tfrac{1}{2}\left[(\hat{c}^{\mathrm{Tri}}_{11}+\hat{c}^{\mathrm{Tri}}_{12}+\hat{c}^{\mathrm{Tri}}_{33}) - \sqrt{8(\hat{c}^{\mathrm{Tri}}_{13})^2+(\hat{c}^{\mathrm{Tri}}_{11}+\hat{c}^{\mathrm{Tri}}_{12}-\hat{c}^{\mathrm{Tri}}_{33})^2}\right],\\
\Lambda^{\mathrm{Tri}}_{(3,6)} &= \tfrac{1}{2}\left[(\hat{c}^{\mathrm{Tri}}_{11}-\hat{c}^{\mathrm{Tri}}_{12}+\hat{c}^{\mathrm{Tri}}_{44}) + \sqrt{8(\hat{c}^{\mathrm{Tri}}_{14})^2+(\hat{c}^{\mathrm{Tri}}_{11}-\hat{c}^{\mathrm{Tri}}_{12}-\hat{c}^{\mathrm{Tri}}_{44})^2}\right],\\
\Lambda^{\mathrm{Tri}}_{(4,5)} &= \tfrac{1}{2}\left[(\hat{c}^{\mathrm{Tri}}_{11}-\hat{c}^{\mathrm{Tri}}_{12}+\hat{c}^{\mathrm{Tri}}_{44}) - \sqrt{8(\hat{c}^{\mathrm{Tri}}_{14})^2+(\hat{c}^{\mathrm{Tri}}_{11}-\hat{c}^{\mathrm{Tri}}_{12}-\hat{c}^{\mathrm{Tri}}_{44})^2}\right],
\end{aligned} \tag{A10}$$

respectively. The first two eigenvalues are the same as those for tetragonal symmetry given by (A6), the associated eigentensors $\mathbf{N}^{(1)\mathrm{Tri}}$ and $\mathbf{N}^{(2)\mathrm{Tri}}$ are given by $\mathbf{N}^{(1)\mathrm{Tet}}$ and $\mathbf{N}^{(2)\mathrm{Tet}}$ in (A7) and the value of the angle α by (A8) with the superscript Tet by the superscript Tri. The eigentensors corresponding to the other eigenvalues are

$$\begin{aligned}
\mathbf{N}^{(3)\mathrm{Tri}} &= \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\beta & 0 & 0\\ 0 & -\cos\beta & \sin\beta\\ 0 & \sin\beta & 0\end{bmatrix}, \mathbf{N}^{(6)\mathrm{Tri}} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & \cos\beta & \sin\beta\\ \cos\beta & 0 & 0\\ \sin\beta & 0 & 0\end{bmatrix},\\
\mathbf{N}^{(4)\mathrm{Tri}} &= \frac{1}{\sqrt{2}}\begin{bmatrix}-\sin\beta & 0 & 0\\ 0 & \sin\beta & \cos\beta\\ 0 & \cos\beta & 0\end{bmatrix}, \mathbf{N}^{(5)\mathrm{Tri}} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -\sin\beta & \cos\beta\\ -\sin\beta & 0 & 0\\ \cos\beta & 0 & 0\end{bmatrix},
\end{aligned} \tag{A11}$$

where

$$\tan 2\beta = \frac{2\sqrt{2}\hat{c}^{\mathrm{Tri}}_{14}}{(\hat{c}^{\mathrm{Tri}}_{11}-\hat{c}^{\mathrm{Tri}}_{12}-\hat{c}^{\mathrm{Tri}}_{44})}. \tag{A12}$$

The eigentensors and the eigenvalues for transversely isotropic or hexagonal symmetry are obtained from those of trigonal symmetry by allowing the elastic constant $\hat{c}^{\mathrm{Tri}}_{14}$ to vanish and replacing the superscript Tri by the superscript Hex. As in the case of trigonal symmetry, there are four distinct eigenvalues for transversely isotropic or hexagonal symmetry. Two are associated with different dilatational modes and two of multiplicity 2 are associated with different isochoric modes. The two dilatational modes, being identical to those of tetragonal or trigonal symmetry, are as described above or in the section on tetragonal symmetry. However, since the elastic constant $\hat{c}^{\mathrm{Tri}}_{14}$ and hence the angle β defined by (A12) vanishes, the third through sixth eigenvalues of (A10) reduce to

$$\Lambda^{\mathrm{Hex}}_{(3,6)} = \hat{c}^{\mathrm{Hex}}_{11} - \hat{c}^{\mathrm{Hex}}_{12}, \qquad \Lambda^{\mathrm{Hex}}_{(4,5)} = \hat{c}^{\mathrm{Hex}}_{44}. \tag{A13}$$

The eigentensors are identical with those given by (A7) for tetragonal symmetry.

The details of the eigenmodes for the triclinic, monoclinic and orthotropic symmetries are not recorded here in detail but may be found in [7] or [10]. Orthotropic symmetry has the three volume preserving shear modes $\mathbf{N}^{(4)\mathrm{Ort}}$, $\mathbf{N}^{(5)\mathrm{Ort}}$ and $\mathbf{N}^{(6)\mathrm{Ort}}$ that are identical to $\mathbf{N}^{(4)\mathrm{Cub}}$, $\mathbf{N}^{(5)\mathrm{Cub}}$ and $\mathbf{N}^{(6)\mathrm{Cub}}$, respectively, given by (A3) as well as three dilatational modes like $\mathbf{N}^{(1)\mathrm{Tet}}$ and $\mathbf{N}^{(2)\mathrm{Tet}}$ in equation (A7). Monoclinic symmetry has the two isochoric modes that are designated as, say, $\mathbf{N}^{(5)\mathrm{Mon}}$ and $\mathbf{N}^{(6)\mathrm{Mon}}$ that are identical with $\mathbf{N}^{(5)\mathrm{Cub}}$ and $\mathbf{N}^{(6)\mathrm{Cub}}$ given by equation (A3), as well as four dilatational modes. Triclinic symmetry has, in general, six dilatational modes, of which nothing can be said.

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